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The probability distribution of a wave at a very large depth within an extended random medium

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Abstract. A plane wave is incident normally onto the boundary of a semi-infinite stationary random medium. The statistical moments of the field variable, both at one and at several points, are calculated when the refractive index of the medium at different points has a joint Gaussian probability distribution and Gaussian power spectrum, and the observer is at a very great depth within the medium. From these moments the probability distributions are calculated. Although the results are simple, the process by which they are obtained is complicated by the presence of multiple scatter. The method uses a perturbation expansion, the terms of which may be interpreted physically as different orders of scattering in a 'successive scattering' approach.

In the limit of infinite distance the single-point probability distribution of the field is a Rayleigh distribution of amplitude together with a uniform distribution of phase. For smaller depths there are situations in which this field has an additive constant component so that the resultant amplitude has a Rice distribution.

In the limit of infinite depth, the moments for two or more points separate into the product of the contributions from the individual points. Thus the fields at different points are statistically independent and their joint probability distribution is simply the product of Gaussian distributions which describe the separate fields. The spatial power spectrum of the observed signal is then constant.

1. Introduction

In considering the propagation of electromagnetic waves in extended random media, it has been long and widely assumed that after a sufficiently great distance has been traversed the varying component of the field attains a Rayleigh distribution about its mean value and the phase of this varying part is uniformly distributed. Under certain circumstances (if the onset of Gaussian statistics occurs before the mean field has decayed to zero) this is centred on some non-zero value, leading to a Rice distribution of field strength or a Rice-squared distribution of intensity. As the distance increases the mean field tends to zero, resulting in a Rayleigh distribution of field strength and an exponential distribution of intensity.

One of the main applications of this work is in radio astronomy, when the propagation of waves through the interplanetary and interstellar media is studied. In this field, work has been done to investigate experimentally the occurrence or otherwise of these distributions: for example Milne (1975) has compared the distributions

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of intensity from small radio sources with three known distributions (the Rice-squared, truncated Gaussian and log-normal distributions) and has shown that the Rice-squared distribution gives the best fit to the experimental data.

In addition a large amount of work has been done under the assumption of Gaussian statistics: Fante (1975) has described a method of calculating the field moments in the case of multiple scatter by starting with their known values in the far-distance (Gaussian) limit and extrapolating toward the source. Budden and Uscinski (1970, 1971, 1972) have assumed Gaussian statistics to calculate scintillation indices for extended sources observed with receivers having finite bandwidth.

Although the onset of Gaussian statistics has been so well known and so well used, there has been until now no rigorous proof that it is correct, though many authors have written on the subject. De Wolf (1975) has reviewed the arguments for a Rayleigh or a Rice distribution of field strength and those for a log-normal distribution. This paper presents a semi-rigorous proof that for sufficiently large distances and for all values of the medium parameters, subject to their obeying rather broad relationships given in the next section, the field has a Rayleigh distribution, while at smaller distances and under certain circumstances it can have a Rice distribution. These results are in agreement with the conditions set down by de Wolf for these distributions, the log-normal distribution occurring at lesser depths. The method is related to that used by Mercier (1962) for the problem of scattering by a phase thick screen.

The contribution to the field observed at a given constant z plane after perturbation by the refractive index fluctuations ('scattering') at m previous planes is calculated. In this way a perturbation series is derived of which the above quantity is the mth term. Between perturbations the wave propagates as though through free space. The moments of its field may then be expressed as infinite series, each term representing the contribution of one order of scattering and splitting into a sum of contributions, each of which may be described by a diagram. The majority of these are shown to be negligible in the limit of infinite depth and the ones that survive are easily calculated. There is a clear correspondence between the diagrams which survive this rejection and those integrals in the paper by Mercier (1962) (around equations (15) and (16)) which do not tend to zero for large z. The probability distribution is then calculated from the moments.

Shishov (1971) has derived expressions for the first and second moments of intensity for a wave propagating in such a medium and has shown that in the limit of large distance these bear the relationship to one another required by a Rayleigh distribution of field strength. We have extended this to show that the relationships between all moments of the field, including those which are not intensity moments, are consistent with the assumption of a Rayleigh distribution and have consequently proved the validity of this assumption.

2. The physical situation

We shall consider a plane wave with wavenumber k incident normally on the boundary $z=0$ of a medium-filled half space. For $z<0$ the refractive index is $n=1$ (free space) and, for $z>0$, n takes the form

$$n = n_0 + n_1 n_2(x, y, z). \tag{1}$$

Here n_0 is the mean value of n, n_1 is its standard deviation and n_2 is a stochastic function of position whose values at different points are assumed to obey a joint-normal probability distribution and a Gaussian autocorrelation function:

$$\rho(\Delta x, \Delta y, \Delta z) \equiv \langle n_2(x, y, z) n_2(x+\Delta x, y+\Delta y, z+\Delta z)\rangle$$
$$= \exp[-(\Delta x^2 + \Delta y^2 + \Delta z^2)/r_0^2] \quad (2)$$

where r_0 is the scale size of the refractive index fluctuations. This form of autocorrelation function is used for mathematical convenience, though there is some evidence (e.g. Rumsey 1975, Chytil 1975) that in many applications, particularly in the ionosphere and upper atmosphere, other forms such as the truncated power-law spectrum are more realistic.

In practice we shall have to simplify (2) further by making the commonly used assumption that successive layers of the medium scatter independently. This is done by replacing (2) by a modified function:

$$\rho'(\Delta \boldsymbol{r}, \Delta z) = r_0 \pi^{1/2} \delta(\Delta z) \exp(-\Delta \boldsymbol{r}^2/r_0^2) \quad (3)$$

where $\Delta \boldsymbol{r}$ is the two-dimensional transverse position vector and the constant factor $r_0 \pi^{1/2}$ has been chosen so that ρ and ρ' have the same integrals over the whole Δz axis. It is emphasised that ρ' is not the true autocorrelation function of n_2, which should be unity at the origin and should in many practical applications be isotropic. However, the effect on the moments of this change is negligible and it does not alter the physics of the problem. This approximation is known variously as the delta-correlation or Markov approximation and a more detailed justification of its use is given, for example, by Barabanenkov *et al* (1971).

In most applications in astronomy $n_0 \approx 1$ and $n_1 \ll 1$. In putting $n_0 = 1$ we are not making any real restriction, since its value affects the whole wavefront equally. It is also usually true in such applications that the scale size r_0 is very large in comparison with wavelength $2\pi/k$ of the radiation. A consequence of this is that negligible amounts of energy are scattered through angles which are not small in comparison with one radian. We shall assume a stronger condition, namely that not only are individual angles of scatter small, but that the total angular deviation acquired on passage through great depths of medium is also small. Of course, this imposes a constraint on the values of z which we can consider: if z is increased indefinitely it must ultimately reach a value where the angle of scatter is not small and where some of the radiation is scattered backwards. It is here assumed that the medium is sufficiently weak for depths to exist which are large enough to allow the subsequent approximations to be made but not so large that large angle scattering is important, and we also neglect any radiation scattered backwards from extremely remote regions beyond the observer. These approximations are valid in many astronomical applications.

The complex exponential notation for the waves is used. Thus the incident wave is $E_{\mathrm{inc}} \exp(\mathrm{i}kz)$ where a time-dependent factor $\exp(-\mathrm{i}\omega t)$ has been suppressed. Within the medium the amplitude and phase of the wave are represented by the modulus and argument of a complex number E.

In order to derive an expression for the general moment

$$R_{\alpha\beta} \equiv \langle E^{\alpha} E^{*\beta} \rangle \quad (4)$$

of the field at a point in the medium, we must first derive an explicit expression, dependent on the particular realisation of the medium, for the field at that point, as a sum of components having suffered 0, 1, 2... scatters.

3. Components of the field

Although the fluctuations in refractive index form a stationary stochastic function of position and consequently do not have a proper Fourier transform, we may use the techniques employed in generalised harmonic analysis (e.g. Kubo 1966, § 3, Reif 1965, §§ 15.13, 15.15) to express this function in terms of spatial frequency components. This may be done, for example, by truncating the function outside a large region and taking the Fourier transform of the truncated function. Although the transform itself does not have a limit as the range tends to infinity, the properties we require do have well defined limits, so we shall write for arbitrarily large X:

$$A(\boldsymbol{\mu}, z)=\frac{1}{4\pi^2}\int_{-X}^{X}\int_{-X}^{X} n_2(\boldsymbol{r}, z)\exp(-\mathrm{i}\boldsymbol{\mu}\cdot\boldsymbol{r})\,\mathrm{d}^2\boldsymbol{r}$$

with (5)

$$n_2(\boldsymbol{r}, z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\boldsymbol{\mu}, z)\exp(\mathrm{i}\boldsymbol{\mu}\cdot\boldsymbol{r})\,\mathrm{d}^2\boldsymbol{\mu}$$

where $-X<x, y<X$ and $\boldsymbol{r}$ is the vector (x, y). In (5) $\boldsymbol{\mu}$ is a spatial frequency vector in the (x, y) plane and A is a complex stochastic quantity having zero mean.

Consider the field at a depth z due to an incident wave that has been scattered just once in a layer of thickness $\mathrm{d}z_1$ at z_1. Immediately after the layer the field is

$$E_{\text{inc}}\exp(\mathrm{i}kz_1)\exp[\mathrm{i}k(1+n_1n_2)\,\mathrm{d}z_1]$$
$$=E_{\text{inc}}\exp[\mathrm{i}k(z_1+\mathrm{d}z_1)]+E_{\text{inc}}\mathrm{i}kn_1n_2\,\mathrm{d}z_1\exp[\mathrm{i}k(z_1+\mathrm{d}z_1)]+\mathrm{O}(\mathrm{d}z_1^2). \quad (6)$$

The first term of (6) is the unscattered field. The component of the field that has been scattered once is the second term and may be written

$$E_{\text{inc}}\mathrm{i}kn_1\,\mathrm{d}z_1\exp[\mathrm{i}k(z_1+\mathrm{d}z_1)]\iint A(\boldsymbol{\mu}, z_1)\exp(\mathrm{i}\boldsymbol{\mu}\cdot\boldsymbol{r})\,\mathrm{d}^2\boldsymbol{\mu} \quad (7)$$

where we have used the transform (5) for n_2.

Equation (7) may be considered as an angular spectrum of plane waves originating in the plane $z=z_1$. In the small angle approximation this becomes for larger values of z

$$E_{\text{inc}}\mathrm{i}kn_1\,\mathrm{d}z_1\exp(\mathrm{i}kz_1)\iint A(\boldsymbol{\mu}, z_1)\exp(\mathrm{i}\boldsymbol{\mu}\cdot\boldsymbol{r})\exp[\mathrm{i}k(z-z_1)(1-\boldsymbol{\mu}^2/2k^2)]\,\mathrm{d}^2\mu$$
$$=E_{\text{inc}}\mathrm{i}kn_1\,\mathrm{d}z_1\exp(\mathrm{i}kz)\iint A(\boldsymbol{\mu}, z_1)\exp[-\mathrm{i}\boldsymbol{\mu}^2(z-z_1)/2k]\,\mathrm{expi}(\boldsymbol{\mu}\cdot\boldsymbol{r})\,\mathrm{d}^2\boldsymbol{\mu} \quad (8)$$

which may be interpreted as follows: the factor i is the phase advance of $\pi/2$ occurring immediately on scattering, $kn_1\,\mathrm{d}z_1$ is the strength of scattering within a layer of thickness $\mathrm{d}z_1$, $\exp(\mathrm{i}\boldsymbol{\mu}\cdot\boldsymbol{r})$ is the phase fluctuation imposed by scattering from the spatial frequency component $\boldsymbol{\mu}$ and

$$\exp\left(-\frac{\mathrm{i}}{2k}\boldsymbol{\mu}^2(z-z_1)\right) \quad (9)$$

is the 'distance effect'. In addition to the immediate $\frac{1}{2}\pi$ phase advance, scattering alters the phase of the wave through this last factor which varies smoothly from unity as the wave propagates away from the scattering layer.

We can evaluate (8) at $z = z_2$ and introduce a second scattering layer there of thickness dz_2. The doubly-scattered component immediately after the layer is:

$$\begin{aligned} &E_{\text{inc}}(ikn_1)^2\, dz_1\, dz_2 \exp(ikz_2) \iint A(\boldsymbol{\mu}_1, z_1) \exp\left(-\frac{i}{2k}(z_2 - z_1)\boldsymbol{\mu}_1^2\right) \\ &\qquad \times \exp(i\boldsymbol{\mu}_1 \cdot \boldsymbol{r})\, d^2\boldsymbol{\mu}_1 \iint A(\boldsymbol{\mu}_2, z_2) \exp(i\boldsymbol{\mu}_2 \cdot \boldsymbol{r})\, d^2\boldsymbol{\mu}_2 \\ &\qquad = E_{\text{inc}}(ikn_1)^2\, dz_1\, dz_2 \exp(ikz_2) \iiiint A(\boldsymbol{\mu}_1, z_1) \\ &\qquad \times \exp\left(-\frac{i}{2k}(z_2 - z_1)\boldsymbol{\mu}_1^2\right) A(\boldsymbol{\mu}_2, z_2) \exp[i(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) \cdot \boldsymbol{r}]\, d^2\boldsymbol{\mu}_1\, d^2\boldsymbol{\mu}_2. \qquad (10) \end{aligned}$$

The integrand of (10) looks like an angular component due to scattering from a spatial frequency vector $\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2$. It is this frequency which determines the distance effect for propagation from z_2 to z. Hence, writing $A(\boldsymbol{\mu}_j, z_j) \equiv A_j$ we have:

$$\begin{aligned} &E_{\text{inc}}(ikn_1)^2\, dz_1\, dz_2 \exp(ikz) \iiiint A_1 \exp\left(-\frac{i}{2k}(z_2 - z_1)\boldsymbol{\mu}_1^2\right) \\ &\qquad \times A_2 \exp\left(-\frac{i}{2k}(z - z_2)(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)^2\right) \exp[i(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) \cdot \boldsymbol{r}]\, d^2\boldsymbol{\mu}_1\, d^2\boldsymbol{\mu}_2. \qquad (11) \end{aligned}$$

We can continue to add scatters indefinitely by this process. We shall at present require only the value of the field at $\boldsymbol{r} = 0$ because the medium is statistically stationary. Then $E(0, 0, z)$ includes a component having scattered m times given by

$$\begin{aligned} &E_{\text{inc}}(ikn_1)^m\, dz_1 \ldots dz_m \exp(ikz) \int \ldots \int A_1 \exp\left(-\frac{i}{2k}(z_2 - z_1)\boldsymbol{\mu}_1^2\right) \\ &\qquad \times A_2 \exp\left(-\frac{i}{2k}(z_3 - z_2)(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)^2\right) A_3 \exp\left(-\frac{i}{2k}(z_4 - z_3)\right. \\ &\qquad \times \left.(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{\mu}_3)^2\right) \times \ldots \times A_m \exp\left(-\frac{i}{2k}(z - z_m)(\boldsymbol{\mu}_1 + \ldots + \boldsymbol{\mu}_m)^2\right) \\ &\qquad \times d^2\boldsymbol{\mu}_1\, d^2\boldsymbol{\mu}_2 \ldots d^2\boldsymbol{\mu}_m \qquad (12) \end{aligned}$$

which may be written

$$\begin{aligned} &E_{\text{inc}}(ikn_1)^m\, dz_1 \ldots dz_m \exp(-ikz) \int \ldots \int A_1 \exp\left(-\frac{i}{2k}(z - z_1)\boldsymbol{\mu}_1^2\right) \\ &\qquad \times A_2 \exp\left(-\frac{i}{2k}(z - z_2)(\boldsymbol{\mu}_2^2 + 2\boldsymbol{\mu}_2 \cdot \boldsymbol{\mu}_1)\right) A_3 \exp\left(-\frac{i}{2k}(z - z_3)\right. \\ &\qquad \times \left.(\boldsymbol{\mu}_3^2 + 2\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2 + 2\boldsymbol{\mu} \cdot \boldsymbol{\mu}_3)\right) \times \ldots \times A_m \exp\left(-\frac{i}{2k}(z - z_m)\right. \\ &\qquad \times \left.(\boldsymbol{\mu}_m^2 + 2\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_m + 2\boldsymbol{\mu}_2 \cdot \boldsymbol{\mu}_m + \ldots + 2\boldsymbol{\mu}_{m-1} \cdot \boldsymbol{\mu}_m)\right) \\ &\qquad \times d^2\boldsymbol{\mu}_1\, d^2\boldsymbol{\mu}_2 \ldots d^2\boldsymbol{\mu}_m. \qquad (13) \end{aligned}$$

To complete our expression for the mth scattered component of $E(0, 0, z)$ we must integrate with respect to $z_1, \ldots, z_m$ over all values from 0 to z, subject to their remaining in the correct order. Thus

$$E_m = E_{\text{inc}}(\mathrm{i}kn_1)^m \exp(\mathrm{i}kz) \int_0^z \int_0^{z_m} \ldots \int_0^{z_2} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} A_1 \ldots \mathrm{d}^2\boldsymbol{\mu}_m \, \mathrm{d}z_1 \, \mathrm{d}z_2 \ldots \mathrm{d}z_m. \tag{14}$$

The total field is the sum of all components:

$$E(0, 0, z) = \sum_{m=0}^{\infty} E_m(0, 0, z) \tag{15}$$

where E_0 represents the unscattered component.

4. Moments

We are now in a position to take powers of the field (15) and take averages to form moments:

$$\begin{aligned} R_{\alpha\beta} &= \langle E^\alpha E^{*\beta} \rangle \\ &= \left\langle \left(\sum_m E_m\right)^\alpha \left(\sum_l E_l^*\right)^\beta \right\rangle \\ &= \sum_{m(1)} \ldots \sum_{m(\alpha)} \sum_{l(1)} \ldots \sum_{l(\beta)} \langle E_{m(1)} \ldots E_{m(\alpha)} E_{l(1)}^* \ldots E_{l(\beta)}^* \rangle \end{aligned} \tag{16}$$

where all summations run from zero to infinity. Write $N_1 = \Sigma_{i=1}^{\alpha} m(i)$ and $N_2 = \Sigma_{j=1}^{\beta} l(j)$. The quantity inside the angular brackets of (16) is a great multi-dimensional integral in which the number of A's is $N_1 + N_2$. It is over the product of these A's that the assembly average is taken, as it is these that contain all the stochastic nature of the fields.

We come now to the central problem of evaluating this average. The value of $\langle A_1 A_2 \ldots A_p A_{p+1}^* \ldots A_{p+q}^* \rangle$ is calculated in appendix 1. It is shown there that : (i) this average is zero if $p+q$ is odd; (ii) it splits into a sum of product of pairs of the form $\langle A_j A_l \rangle$ or $\langle A_j^* A_l^* \rangle$ or $\langle A_j A_l^* \rangle$ if $p+q$ is even; and (iii) the values of these three autocorrelation functions are

$$\tfrac{1}{4} r_0^3 \pi^{-1/2} \delta^2(\boldsymbol{\mu}_j \pm \boldsymbol{\mu}_l) \delta(z_j - z_l) \exp(-\tfrac{1}{4} \boldsymbol{\mu}_j^2 r_0^2), \tag{17}$$

the plus sign being taken in the first two cases and the minus sign in the third. In deriving (17) we have used the delta-correlation approximation (3).

The two delta functions in (17) tell us that for two scatters to be correlated they must occur at the same depth and from spatial frequency vectors that are equal and antiparallel for $\langle A_j A_l \rangle$ and $\langle A_j^* A_l^* \rangle$, and are equal and parallel for $\langle A_j A_l^* \rangle$.

Thus $N_1 + N_2$ must be even, say $N_1 + N_2 = 2N$. N is the order of the term.

5. The diagram technique

It is convenient at this stage to introduce a technique for describing the various terms of (16) by means of diagrams. Each term is represented by a set of diagrams containing

$\alpha+\beta$ horizontal lines, each of which represents one of the fields $E_{m(i)}$ or $E^*_{l(j)}$. A scatter occurring in one of the fields is denoted by a dot on the corresponding line, in a position along the line given roughly by the value of z at which the scatter occurs. Since only correlated pairs of scatters have any significance, we may join such pairs by lines, vertical if the dots are on different horizontal lines or hooked if they are on the same line, to show the correlations. The order of these lines from the left represents the order of occurrence of the scatter pairs. Figure 1 represents three of the contributions to $\langle E_2E_2E_2E_1E_2^*E_1^*E_0^*\rangle$ in which the fields are labelled from top to bottom.

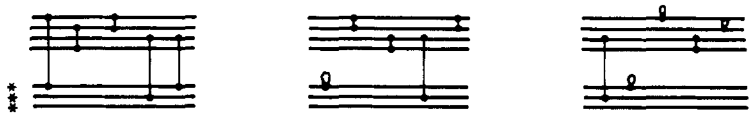

Figure 1. Three contributions to the term $\langle E_2E_2E_2E_1E_2^*E_1^*E_0^*\rangle$.

Each pair of scatters is of one of three types:

(i) it may involve one starred and one unstarred field;

(ii) it may involve either two starred or two unstarred fields; or

(iii) it may involve only one field.

Each diagram represents: (a) a $2N$-fold integral over the z_i; and (b) a $4N$-fold integral over the $\boldsymbol{\mu}_i$. N of the integrals (a) and $2N$ of the integrals (b) may be performed immediately on account of the delta functions. We can label the scatter pairs by the order of the depths at which they occur and then integrate over these depths keeping them in order as in (14).

The problem of 'double scatters', that is, scatter pairs of type (iii) above, is dealt with in appendix 2. It is shown there that their presence affects the contribution made by a diagram to the moment by an overall factor dependent only on the medium parameters, the depth z and the order of the moment, and hence outside the integrals. The effect of all possible configurations of double scatters in otherwise identical diagrams is included simply by multiplying the contribution of the basic diagram (without any double scatters) by

$$\exp[-\tfrac{1}{2}k^2n_1^2r_0\pi^{1/2}(\alpha+\beta)z] = \exp[-\tfrac{1}{2}(\alpha+\beta)l] \tag{18}$$

where

$$l = k^2n_1^2r_0\pi^{1/2}z.$$

Now we need only consider diagrams containing scatter pairs of types (i) and (ii).

6. Evaluating the diagrammatic contributions and selecting the dominant ones

The integrand of (16) has become the exponential of a quadratic form in the vectors $\boldsymbol{\mu}_j$:

$$\exp(-M_{jl}\boldsymbol{\mu}_j\cdot\boldsymbol{\mu}_l) = \exp(-\mu_j^{(x)}M_{jl}\mu_l^{(x)} - \mu_j^{(y)}M_{jl}\mu_l^{(y)}) \tag{19}$$

where $\boldsymbol{\mu}_j = (\mu_j^{(x)}, \mu_j^{(y)})$ and the summation convention has been used. The jth diagonal element of the symmetric $N\times N$ matrix $\mathbf{M}$ is of the form

$$M_{jj} = \tfrac{1}{4}r_0^2 + \mathrm{i}\alpha_{jj}\frac{z-z_j}{2k} = \tfrac{1}{4}r_0^2(1+2\mathrm{i}\alpha_{jj}\zeta y_j) \tag{20}$$

where

$$y_j = (z - z_j)/z \tag{21}$$

$$\zeta = z/kr_0^2 \tag{22}$$

and α_{jj} is some integer dependent on the particular diagram in question. The first term of (20) arises from the Gaussian factor in (17) and the second from the relevant distance effect factor (9). The integer α_{jj} lies between -2 and $+2$, and we shall later examine the conditions under which it is zero, these being the only properties we require.

The reason for the changes of variable (21) and (22) is that the y_j all lie within the range (0, 1) and so do not get large with z, and that z/kr_0^2 is the natural way of scaling z, being its value in units of the Fresnel length.

The (j, l) and (l, j) off-diagonal elements of $\mathbf{M}$ with $j < l$ are of the form

$$M_{jl} = M_{lj} = \tfrac{1}{4}r_0^2 \times 2\mathrm{i}\alpha_{jl}\zeta y_l \tag{23}$$

and the remarks concerning α_{jj} apply also to α_{jl}.

The remaining $\boldsymbol{\mu}_j$ integrals may now be evaluated in the usual way, by transforming the above quadratic form into a system of coordinates in which $\mathbf{M}$ is diagonal. The integrals over the components $\mu_j^{(x)}$ and $\mu_j^{(y)}$ of the vectors $\boldsymbol{\mu}_j$ produce identical factors and the result of the integrations is

$$\pi^N/\det(\mathbf{M}). \tag{24}$$

The contribution made by any diagram to the moment $R_{\alpha\beta}$ is of the form (using (14), (16), (18) and (24))

$$E_{\mathrm{inc}}^{\alpha}E_{\mathrm{inc}}^{*\beta}(\mathrm{i}kn_1)^{N_1}(-\mathrm{i}kn_1)^{N_2}\exp[\mathrm{i}k(\alpha-\beta)z]\exp[-\tfrac{1}{2}(\alpha+\beta)l]$$
$$\times\left(\frac{\pi r_0^3}{4\pi^{1/2}}\right)^N \int_0^1\int_{y_N}^1\int_{y_{N-1}}^1\cdots\int_{y_2}^1 \frac{\mathrm{d}y_1\mathrm{d}y_2\ldots\mathrm{d}y_m}{\det(\mathbf{M})}. \tag{25}$$

As an example of how this works in practice, consider the diagram in figure 2. After the $\boldsymbol{\mu}$'s and z's in each scatter pair have been equated by integration of the delta functions, the exponent of the integral corresponding to this diagram is

$$\begin{array}{ll}
-\dfrac{\mathrm{i}}{2k}[(z_5-z_3)\boldsymbol{\mu}_3^2+(z-z_5)(\boldsymbol{\mu}_3+\boldsymbol{\mu}_5)^2 & \text{from 1st line}\\
+(z_6-z_2)\boldsymbol{\mu}_2^2+(z-z_6)(\boldsymbol{\mu}_2+\boldsymbol{\mu}_6)^2 & \text{from 2nd line}\\
+(z_4-z_1)\boldsymbol{\mu}_1^2+(z-z_4)(\boldsymbol{\mu}_1+\boldsymbol{\mu}_4)^2 & \text{from 3rd line}\\
+(z_6-z_2)(-\boldsymbol{\mu}_2)^2+(z-z_6)(-\boldsymbol{\mu}_2-\boldsymbol{\mu}_6)^2 & \text{from 5th line}\\
-(z-z_5)\boldsymbol{\mu}_5^2 & \text{from 6th line}\\
-(z_4-z_1)\boldsymbol{\mu}_1^2+(z-z_4)(\boldsymbol{\mu}_1+\boldsymbol{\mu}_4)^2 & \text{from 7th line}\\
-(z-z_3)\boldsymbol{\mu}_3^2] & \text{from 8th line}\\
-\tfrac{1}{4}r_0^2(\boldsymbol{\mu}_1^2+\boldsymbol{\mu}_2^2+\boldsymbol{\mu}_3^2+\boldsymbol{\mu}_4^2+\boldsymbol{\mu}_5^2+\boldsymbol{\mu}_6^2) & \text{from correlations as in (17)}
\end{array} \tag{26}$$

$$=\tfrac{1}{4}r_0^2\{-2\mathrm{i}\zeta[2\boldsymbol{\mu}_3\cdot\boldsymbol{\mu}_5y_5+2\boldsymbol{\mu}_2^2y_2+(2\boldsymbol{\mu}_6^2+4\boldsymbol{\mu}_2\cdot\boldsymbol{\mu}_6)y_6]$$
$$-\boldsymbol{\mu}_1^2-\boldsymbol{\mu}_2^2-\boldsymbol{\mu}_3^2-\boldsymbol{\mu}_4^2-\boldsymbol{\mu}_5^2-\boldsymbol{\mu}_6^2\}. \tag{27}$$

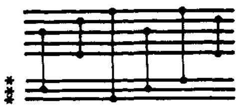

Figure 2. A sixth-order contribution to the moment $\langle E^5E^{*3}\rangle$.

The matrix $\mathbf{M}$ is given by:

$$\mathbf{M}=\tfrac{1}{4}r_0^2(\mathbf{I}+2\mathrm{i}\zeta\mathbf{M}') \tag{28}$$

where $\mathbf{I}$ is the sixth order unit matrix and $\mathbf{M}'$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2y_2 & 0 & 0 & 0 & 2y_6 \\ 0 & 0 & 0 & 0 & y_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_5 & 0 & 0 & 0 \\ 0 & 2y_6 & 0 & 0 & 0 & 2y_6 \end{pmatrix}. \tag{29}$$

In general (28) will hold with $\mathbf{M}'$ given by

$$\begin{aligned} M'_{jj} &= \alpha_{jj}y_j \\ M'_{jl} &= M'_{lj} = \alpha_{jl}y_l \qquad (l>j). \end{aligned} \tag{30}$$

Now any non-zero α_{jl} is enough to alter the value of the determinant $\mathbf{M}$ from the value of

$$(\tfrac{1}{4}r_0^2)^N \tag{31}$$

which it would take if all the α_{jl} were zero. For example, if α_{jj} were non-zero, det($\mathbf{M}$) would contain a term

$$(\tfrac{1}{4}r_0^2)^N 2\mathrm{i}\zeta\alpha_{jj}y_j \tag{32}$$

while if $\alpha_{jl}(j<l)$ were non-zero, det($\mathbf{M}$) would contain a term

$$-(\tfrac{1}{4}r_0^2)^N(2\mathrm{i}\zeta\alpha_{jl}y_l)^2. \tag{33}$$

For large ζ the new term will dominate the term (31) and the contribution (25) will be $O(\zeta^{-1})$ in comparison with contributions for which all the $\alpha_{jl}=0$. Thus we may neglect all diagrams that give rise to non-zero $\mathbf{M}'$ provided we prove that for any N there exists at least one diagram having zero $\mathbf{M}'$. Such a diagram will be called a dominant diagram.

We must now look back to expressions (13) and (17) to see which kinds of diagrams are dominant. It is clear that if any pair of scatters is of type (ii) each of the two scatters in the pair will contribute an equal factor

$$\exp\left(\mp\mathrm{i}\frac{(z-z_j)}{2k}\boldsymbol{\mu}_j^2\right) \tag{34}$$

to the integrand and the corresponding matrix will have $\alpha_{jj}=\pm2$. The determinant will thus not be $(\frac{1}{4}r_0^2)^N$ and the diagram non-dominant. A necessary condition for a diagram to be dominant is thus that all its scatter pairs be of type (i). In diagrammatic terms, all vertical lines must link a starred to an unstarred field.

Further, if any field undergoes two scatters, each of which is correlated with one in different fields as in figure 3, then that field will introduce into the integrand a factor

$$\exp\left(\mp i\frac{(z-z_l)}{2k}2\boldsymbol{\mu}_j\cdot\boldsymbol{\mu}_l\right) \tag{35}$$

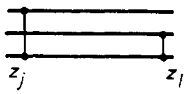

Figure 3. A possible feature of a non-dominant diagram.

and since $\boldsymbol{\mu}_j\cdot\boldsymbol{\mu}_l$ dependence can only arise from a field involved in both the jth and lth scatter pairs, of which there is only the one, $\alpha_{jl}=\pm1$. The diagram is thus non-dominant. If, however, the jth and lth scatter pairs involve the same two fields, one starred and one unstarred, each field introduces a factor in $\boldsymbol{\mu}_j\cdot\boldsymbol{\mu}_l$. These two factors will be complex conjugates of one another and their exponents will cancel. Thus $\alpha_{jl}=0$.

A second necessary condition for a diagram to be dominant is thus that each field may be involved in correlated scatter pairs with only one other field, though there may be any number of pairs of scatters linking them. In diagrammatic terms, each horizontal line of a diagram must be linked by vertical lines to only one other line which must be of opposite type (starred or unstarred) to itself. It may be so linked by an arbitrary number of links.

These two conditions are together sufficient to ensure that a diagram is dominant. That this is so may be seen by arguments along the same lines as those for their necessity; if each field has a twin of the opposite sort, which undergoes scatters from the same spatial frequency components at the same depths, then the two fields will introduce equal and opposite phases to the integrand through the distance effect factors (9). The matrix will thus be simply $\frac{1}{4}r_0^2\mathbf{I}$ and the diagram dominant.

In the example, in figure 2, the first and fourth scatter pairs, linking the third and seventh fields, could be part of a dominant diagram since the third field is unstarred and the seventh starred, and each is linked only to the other. This shows in the matrix $\mathbf{M}'$ ((29)) as zeros filling the first and fourth rows and columns. The second and sixth scatters violate the first criterion for dominance, and the third and fifth violate the second. These violations appear in (29) as non-zero elements in the second, third, fifth and sixth rows and columns. A diagram is dominant only if all its scatters obey the criteria and hence $\mathbf{M}'$ is zero. Examples of dominant and non-dominant diagrams are shown in figures 4(a) and 4(b) respectively.

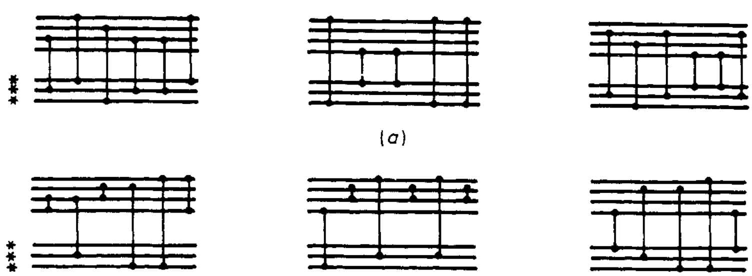

Figure 4. (*a*) Some dominant diagrams: (*b*) non-dominant diagrams.

Clearly for any N such a diagram exists provided α and β are both non-zero. We shall consider the problem of $\langle E^\alpha\rangle$ later.

The above conditions rule out the majority of possible diagrams of any order N, and the proportion excluded increases with N. It may be asked at this point whether the larger number of non-dominant diagrams may not outweigh their smallness so that their resultant contribution is comparable to, or greater than that of the dominant diagrams. This question is discussed briefly in appendix 3.

7. Evaluation of the contribution from dominant diagrams

The fact that in a dominant diagram different pairs of fields are not linked means that the contribution from such diagrams factorises into expressions characteristic of first or second moment diagrams. This is shown schematically in figure 5, and is of crucial importance because, when we sum over diagrams, we will obtain a sum of products of first and second moments similar to the right-hand side of equation (A.2) in appendix 1, suggesting a Gaussian distribution of the field.

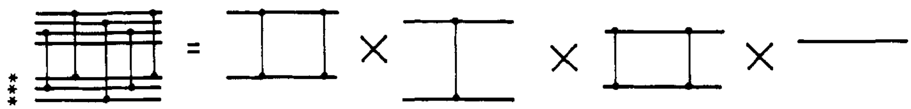

Figure 5. A schematic demonstration of the factorisation of a dominant diagram.

Ascribe a label (a, b) to a diagram if the ath and bth field lines are linked by one or more scatters. In general a diagram will possess several such labels; the diagram in figure 5 for example has labels (1, 5), (2, 7) and (3, 6). Now if we sum over all diagrams having a given set of j labels and no others ($j \leqslant \min(\alpha, \beta)$) we obtain:

$$\langle (E - E_0\,e^{-l/2})(E - E_0\,e^{-l/2})^*\rangle^j \langle E_0\,e^{-l/2}\rangle^{\alpha-j} \langle E_0^*\,e^{-l/2}\rangle^{\beta-j}$$
$$= (\langle EE^*\rangle - \langle E_0E_0^*\rangle\,e^{-l})^j\,\langle E_0\rangle^{\alpha-j}\langle E_0^*\rangle^{\beta-j}\,e^{-\frac{1}{2}(\alpha+\beta-2j)l}. \qquad (36)$$

The second moments here are $\langle (E - E_0\,e^{-l/2})(E - E_0\,e^{-l/2})^*\rangle$ rather than $\langle EE^*\rangle$ because in our definition of the label (a, b) we have excluded the case where the ath and bth fields have no scatters. In this case both ath and bth fields enter the 'pool' of unscattered fields and the diagram has $j-1$ labels. Now we know that:

$$\begin{aligned}\langle E_0\rangle &= E_{\mathrm{inc}}\exp(\mathrm{i}kz)\\ \langle E_0E_0^*\rangle &= |E_{\mathrm{inc}}|^2\\ \langle EE^*\rangle &= |E_{\mathrm{inc}}|^2.\end{aligned} \qquad (37)$$

The first two of these follow immediately from (14) and the third is a statement of the law of conservation of energy. As a demonstration of the use of the diagrammatic technique, the third result may alternatively be derived by summing the second moment diagrams shown in figure 6. The mth order component is easily shown to be

$$|E_{\mathrm{inc}}|^2\exp(-l)l^m/m! \qquad (38)$$

and this is summed over m to give the desired result.

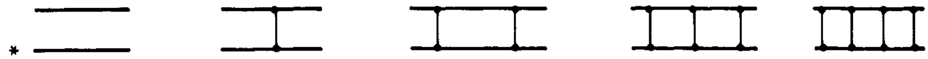

Figure 6. The contributions to the moment $\langle EE^* \rangle$.

Insertion of (37) in (36) gives

$$E^{\alpha}_{\mathrm{inc}}E^{*\beta}_{\mathrm{inc}}[1-\exp(-l)]^{j}\exp[-\tfrac{1}{2}(\alpha+\beta-2j)l]\exp[i(\alpha-\beta)kz]. \tag{39}$$

This result must be multiplied by the number of ways of choosing j pairs of fields, one of each pair being taken without replacement from a set of α fields and the other from an independent set of β fields. To get a complete answer for the moment $R_{\alpha\beta}$ this result is then summed over all possible j's:

$$\begin{aligned} R_{\alpha\beta} = \sum_{j=0}^{\min(\alpha,\beta)} {}_{\alpha}C_j \times {}_{\beta}C_j \times j!\,E^{\alpha}_{\mathrm{inc}}E^{*\beta}_{\mathrm{inc}}[1-\exp(-l)]^{j}\exp[-\tfrac{1}{2}(\alpha+\beta-2j)l] \\ \times\exp[\mathrm{i}kz(\alpha-\beta)]. \end{aligned} \tag{40}$$

l is proportional to z and so, in the limit of large z, we need only consider terms for which $\alpha+\beta-2j$ is zero. Since j is less than or equal to the smaller of α and β, this condition will never be met unless $\alpha=\beta$. Hence

$$R_{\alpha\beta}=\alpha!|E_{\mathrm{inc}}|^{2\alpha}\delta_{\alpha\beta}. \tag{41}$$

This is our final answer for the moment $R_{\alpha\beta}$ in the limit of infinite depth within the medium, provided α and β are both non-zero. If both are zero the moment is trivially unity and (41) still holds. If one is zero then there can be no dominant diagrams. Every diagonal element of each matrix $\mathbf{M}'$ is non-zero and so for large ζ the contribution looks like ζ^{-N}. We can neglect every contribution in comparison with the zero order one, which itself decays as $\exp(-\frac{1}{2}\alpha l)$. Thus (41) still applies and is good for all cases.

8. The probability distribution

Since knowledge of all the moments of a statistical quantity is equivalent to knowledge of its probability distribution, we may construct the two-dimensional distribution $P(E_{\mathrm{R}}, E_{\mathrm{I}})$ from the moments (41). Here E_{R} and E_{I} are the components of E in the Argand diagram.

The characteristic function is

$$\begin{aligned} C(U) &= \langle\exp[\mathrm{i}(U_{\mathrm{R}}E_{\mathrm{R}}+U_{\mathrm{I}}E_{\mathrm{I}})]\rangle = \langle\exp[\tfrac{1}{2}\mathrm{i}(U^*E+E^*U)]\rangle \\ &= \sum_{j=0}^{\infty}(\tfrac{1}{2}\mathrm{i})^{j}\frac{\langle(U^*E+E^*U)^{j}\rangle}{j!} \\ &= \sum_{j=0}^{\infty}(\tfrac{1}{2}\mathrm{i})^{j}\frac{1}{j!}\sum_{k=0}^{j}{}_{j}C_{k}U^{*k}U^{j-k}\langle E^{k}E^{*j-k}\rangle. \end{aligned} \tag{42}$$

$\langle E^{k}E^{*j-k}\rangle$ is zero unless $k=j-k$, which can only be satisfied for even j. Reversing the order of summations and putting $j=2k$ gives:

$$C(U)=\sum_{k=0}^{\infty}(\tfrac{1}{2}\mathrm{i})^{2k}\frac{1}{(2k)!}{}_{2k}C_k|U|^{2k}|E_{\mathrm{inc}}|^{2k}k!$$

$$=\sum_{k=0}^{\infty}(-\tfrac{1}{4}|U|^2|E_{\mathrm{inc}}|^2)^k\frac{1}{k!}=\exp(-\tfrac{1}{4}|U|^2|E_{\mathrm{inc}}|^2). \quad (43)$$

$P(E_{\mathrm{R}}, E_{\mathrm{I}})$ is now obtained by reversing the Fourier transform (42):

$$P(E_{\mathrm{R}}, E_{\mathrm{I}})=\frac{1}{2\pi|E_{\mathrm{inc}}|^2}\exp\left(-\frac{|E|^2}{|E_{\mathrm{inc}}|^2}\right). \quad (44)$$

Hence the two components of E in the Argand diagram obey a two-dimensional uncorrelated Gaussian probability distribution, each component having zero mean and variance equal to half the square of the incident amplitude.

9. The Rice distribution

In deriving the probability distribution (44) we assumed that $\zeta\equiv z/kr_0^2$ was large compared with unity in order to reach expression (36). We later assumed that l was large compared with unity after equation (40). Both these parameters are scaled versions of z and for a given medium both conditions are satisfied in the limit $z\to\infty$ subject to the considerations of § 2. If z is large but finite, it is possible that one or other will not be satisfied.

We can make no progress at all without assuming that ζ is large: the elimination of the large part of the diagrams is the central feature of our method. On the other hand, it is not necessary to assume that l is large in order to get some answers. In this case (40) becomes

$$R_{\alpha\beta}=E_{\mathrm{inc}}^{\alpha}E_{\mathrm{inc}}^{*\beta}\exp[\mathrm{i}(\alpha-\beta)kz]\exp[-\tfrac{1}{2}(\alpha+\beta)l]\sum_{j=0}^{\min(\alpha,\beta)}{}_{\alpha}C_{j\,\beta}C_j\,j![\exp(l)-1]^j \quad (45)$$

which is expressible either in confluent hypergeometric functions or in Laguerre polynomals:

$$R_{\alpha\beta}=E_{\mathrm{inc}}^{\alpha}E_{\mathrm{inc}}^{*\beta}\exp[\mathrm{i}(\alpha-\beta)kz]\exp[-\tfrac{1}{2}(\alpha+\beta)l]\frac{\alpha!}{(\alpha-\beta)!}$$
$$\times[\exp(l)-1]^{\beta}{}_1F_1\{-\beta;\alpha-\beta+1;-1/[\exp(l)-1]\} \quad (46)$$

or

$$R_{\alpha\beta}=E_{\mathrm{inc}}^{\alpha}E_{\mathrm{inc}}^{*\beta}\exp[\mathrm{i}(\alpha-\beta)kz]\exp[-\tfrac{1}{2}(\alpha+\beta)l]\beta!$$
$$\times[\exp(l)-1]^{\beta}L_{\beta}^{(\alpha-\beta)}\{-1/[\exp(l)-1]\}. \quad (47)$$

In (46) and (47) we have taken $\beta\leqslant\alpha$ without loss of generality.

These moments may be shown to be those of a complex quantity whose two components obey identical, independent Gaussian probability distributions about non-zero means.

In our case

$$\langle E\rangle=E_{\mathrm{inc}}\exp(\mathrm{i}kz)\exp(-\tfrac{1}{2}l) \quad (48)$$

and the common variance of the two Gaussian distributions is

$$\sigma^2 \equiv \tfrac{1}{2}|E_{\text{inc}}|^2[1-\exp(-l)]. \tag{49}$$

In this case the magnitude $|E|$ of the field has the distribution discussed by Rice (1944, 1945, § 3.10):

$$P(E)=\frac{E}{\sigma^2}\exp\left(-\frac{E^2+E_0^2}{2\sigma^2}\right)I_0\left(\frac{EE_0}{\sigma^2}\right) \tag{50}$$

(46) reduces to the expression derived by Rice for the moments of (50) if we put $\alpha=\beta$.

10. Depths at which these arguments become valid

The previous discussion suggests that the field will have a Rayleigh distribution if ζ and l are both large in comparison with unity, and a Rice distribution if ζ but not l is large. This is an oversimplification, however. For large l the considerations of appendix 3 must be taken into account and complicate the situation. All that can be seen from these arguments is that condition for Rayleigh distribution will involve the relationship between l and ζ, not just their individual values.

Arguments from ray theory (Hannay, private comunication) suggest that the conditions for the Rayleigh distribution are that $l \gg 1$ and $\zeta \gg G^{-1/3}$ where

$$G=k^3r_0^3n_1^2\pi^{1/2}=l/\zeta. \tag{51}$$

The ray arguments also suggest a connection between this distance and the position of the peak in the curve of second intensity moment against depth. This agrees with other arguments (Shishov 1971) implying that such a peak occurs at a value of ζ proportional to $G^{-1/3}$.

For situations in which l is not large, where we expect a Rice distribution, this complication does not arise. As shown in appendix 3, l is a measure of the mean number of scatters and for $l\sim 1$ we need only consider a small number of them. Thus the number of diagrams is small and there is no question of large numbers outweighing small contributions. In this case the condition $\zeta \gg 1$ seems sufficient to ensure a Rice distribution of field strength.

11. The general field moment at several points

The probability distribution (44) contains all the information which we may require about the value of the field at one point. In order to have equally general information about the values of fields at different points, such as correlation functions and so on it is necessary to know the joint probability distribution of the fields at an arbitrary number of points. We will continue to consider only a single frequency characterised by the wavenumber k.

The most general moment of $E(x, y, z)$ at m different points is of the form

$$\langle E^{n_1}(\boldsymbol{r}_1)E^{*p_1}(\boldsymbol{r}_1)E^{n_2}(\boldsymbol{r}_2)E^{*p_2}(\boldsymbol{r}_2)\ldots E^{n_m}(\boldsymbol{r}_m)E^{*p_m}(\boldsymbol{r}_m)\rangle. \tag{52}$$

This may be expanded in diagrams having $\Sigma_j n_j + \Sigma_l p_l$ horizontal lines divided into $2m$ groups as in figure 7. The contribution of each diagram is calculated in the normal way.

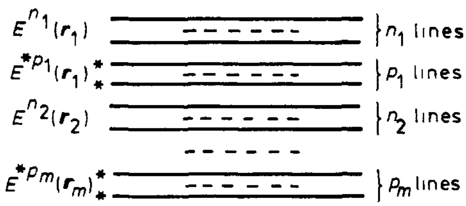

Figure 7. The basic structure of the diagrams contributing to the moment. $\langle E^{n_1}(\boldsymbol{r}_1)E^{*p_1}(\boldsymbol{r}_1)E^{n_2}(\boldsymbol{r}_2)E^{*p_2}(\boldsymbol{r}_2)\ldots E^{*p_m}(\boldsymbol{r}_m)\rangle$.

In fact, the integrand arising from each diagram is simply its value for $\boldsymbol{r}_1 = \boldsymbol{r}_2 = \ldots = \boldsymbol{r}_m = 0$ multiplied by a factor

$$\exp\left(\mathrm{i}\sum(\pm\boldsymbol{\mu}\,.\,\boldsymbol{r})\right). \tag{53}$$

Here the summation is taken over all scatters, $\boldsymbol{\mu}$ is the spatial frequency vector from which the scatter occurs and $\boldsymbol{r}$ is the point at which the relevant field is measured in the observer's plane. The plus or minus sign is taken depending on whether or not the field is starred in (52). (53) arises from factors such as the last factor of (11).

The contribution to (53) from double scatters is zero since both scatters are associated with the same $\boldsymbol{r}$ and opposite $\boldsymbol{\mu}$'s. After dealing with these, each $\boldsymbol{\mu}$ is associated with two fields. If a scatter pair is of type (i), the $\boldsymbol{\mu}$'s associated with each scatter will be parallel, but will have opposite signs in (53), while in the case of a scatter pair of type (ii) the $\boldsymbol{\mu}$'s will be antiparallel and have the same sign in (53). In either case the contribution to the sum will be $\boldsymbol{\mu}\,.\,\Delta\boldsymbol{r}$ where $\Delta\boldsymbol{r}$ is the difference in position between the points of measurements. $\Delta\boldsymbol{r}$ is defined only up to a change of sign but this will not matter since the answer involves only its square. (53) becomes:

$$\exp\left(\mathrm{i}\sum\boldsymbol{\mu}\,.\,\Delta\boldsymbol{r}\right) \tag{54}$$

where the summation is taken over all scatter pairs. The integrand is

$$\exp[-(\mu_j^{(x)}M_{jl}\mu_l^{(x)} + \mu_j^{(y)}M_{jl}\mu_l^{(y)}) + \mathrm{i}(\mu_j^{(x)}\Delta r_j^{(x)} + \mu_j^{(y)}\Delta r_j^{(y)})] \tag{55}$$

from (19) and (54). The exponent now contains terms linear in $\boldsymbol{\mu}_j$ as well as quadratic terms. The integrals with respect to $\boldsymbol{\mu}_j$ can be performed by diagonalising $\mathbf{M}$ as before and then completing the square in $\mu_j^{(x)}$ and $\mu_j^{(y)}$. It may be shown that the result is

$$\frac{\pi^N}{\det(\mathbf{M})}\exp(-\tfrac{1}{4}M_{lm}^{-1}\,\Delta\boldsymbol{r}_l\,.\,\Delta\boldsymbol{r}_m). \tag{56}$$

It is plausible that each element of $\mathbf{M}^{-1}$ should tend to a finite limit as z tends to infinity. That this is so is proved in appendix 4. Hence the exponential in (56) becomes independent of z as z gets large, a diagram has the same order in powers of z as its single-point counterpart and the same two conditions are required in order that it will be dominant. If these are satisfied, $\mathbf{M} = (r_0^2/4)\mathbf{I}$, $\mathbf{M}^{-1} = (4/r_0^2)\mathbf{I}$ and (56) becomes

$$(4\pi/r_0^2)^N\exp\left(-\sum\Delta r_l^2/r_0^2\right) \tag{57}$$

the summation being once more over all pairs of scatters.

Define labels (a, b) as before. Sum over all diagrams having a given set of labels. Then every pair of fields contributes a factor

$$|E_{\text{inc}}|^2 \sum_{n=1}^{\infty} \exp(-l) l^n \exp(-n\Delta r^2/r_0^2)/n! \tag{58}$$

where n is the number of links. This is

$$|E_{\text{inc}}|^2 [\![\exp\{-l[1-\exp(-\Delta r^2/r_0^2)]\} - \exp(-l)]\!]. \tag{59}$$

This tends to zero as l tends to infinity unless $\Delta r = 0$ in which case it is $|E_{\text{inc}}|^2$. Thus, any set of labels gives zero contribution in the limit $z \to \infty$ unless all pairings are between fields measured at the same point. The moment (52) thus splits into a product

$$\langle E^{n_1} E^{*p_1} \rangle \langle E^{n_2} E^{*p_2} \rangle \ldots \langle E^{n_m} E^{*p_m} \rangle. \tag{60}$$

Since this true for all n_j and p_l, the joint $2m$-dimensional probability distribution of the values of the field at m points factorises into m identical two-dimensional single-point distributions which are Gaussian distributions (44). The result tells us that if we go sufficiently far into the medium, the correlation length of the field tends to zero and the values of the field at two points, however close, are independent. This is true in the limit of extreme paraxiality, when the wavelength is vanishingly small in comparison with the transverse scale size r_0 of the irregularities. For finite wavelengths it may be shown by simple arguments based on the wave equation that coherence over a scale of order of one wavelength will be present at any depth. However, if our small angle scattering approximation is to be valid, this will be very short in comparison with all other transverse length scales in the problem.

12. Conclusions

For given medium parameters the probability distribution of the field tends to a two-dimensional circularly-symmetrical Gaussian distribution in the Argand plane centred on the origin, as the observer depth tends to infinity. The variance of the field strength is equal to the square of the incident amplitude. The fields at different points are completely uncorrelated. This is a reasonable approximation provided

$$\zeta \gg G^{-1/3} \qquad \text{or} \qquad z \gg r_0(n_1^2 \pi^{1/2})^{-1/3}. \tag{61}$$

If we fix the value of $l \equiv k^2 n_1^2 r_0 \pi^{1/2} \sim 1$ and consider the limit as $\zeta \to \infty$ subject to this constraint we obtain a two-dimensional circularly-symmetrical Gaussian distribution about a mean

$$E_{\text{inc}} \exp(\mathrm{i}kz) \exp(-\tfrac{1}{2}l) \tag{62}$$

with a variance

$$[1-\exp(-l)]|E_{\text{inc}}|^2. \tag{63}$$

This leads to a Rice distribution of field strength and is a reasonable approximation provided

$$\zeta \gg 1 \qquad \text{or} \qquad z \gg kr_0^2. \tag{64}$$

Acknowledgments

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Appendix 1. The values of the moments of A

Equation (5) expresses the spectrum A in terms of the spatial variations in refractive index. The moment $\langle A_1 \ldots A_p A^*_{p+1} \ldots A^*_{p+q}\rangle$ may be written

$$\frac{1}{(4\pi^2)^{p+q}} \int \ldots \int \langle n_2(\boldsymbol{r}_1, z_1) \ldots n_2(\boldsymbol{r}_{p+q}, z_{p+q})\rangle \exp[-\mathrm{i}(\boldsymbol{r}_1 \cdot \boldsymbol{\mu}_1 + \boldsymbol{r}_2 \cdot \boldsymbol{\mu}_2 + \ldots + \boldsymbol{r}_p \cdot \boldsymbol{\mu}_p - \boldsymbol{r}_{p+1} \cdot \boldsymbol{\mu}_{p+1} - \ldots - \boldsymbol{r}_{p+q} \cdot \boldsymbol{\mu}_{p+q})] \mathrm{d}^2\boldsymbol{r}_1 \ldots \mathrm{d}^2\boldsymbol{r}_{p+q} \tag{A.1}$$

Since the values of n_2 at different points have a joint-normal distribution, the average over $p+q$ values of A is zero if $p+q$ is odd, and splits into the sum of all possible factorisations into pairs, each pair being averaged independently if $p+q$ is even. For example, if P, Q, R and S are four random variables having a joint-normal distribution about zero means, then

$$\langle PQRS\rangle = \langle PQ\rangle\langle RS\rangle + \langle PR\rangle\langle QS\rangle + \langle PS\rangle\langle QR\rangle. \tag{A.2}$$

Using this result we may split expression (A.1) into a sum of integrals in which the A's are paired. These integrals factorise into integrals of the form

$$\langle A_j A_l\rangle = \frac{1}{(4\pi^2)^2} \int \ldots \int \langle n_2(\boldsymbol{r}_j, z_j) n_2(\boldsymbol{r}_l, z_l)\rangle \exp[-\mathrm{i}(\boldsymbol{r}_j \cdot \mu_j + \boldsymbol{r}_l \cdot \boldsymbol{\mu}_l)]\, \mathrm{d}^2\boldsymbol{r}_j\, \mathrm{d}^2\boldsymbol{r}_l \tag{A.3}$$

and three other integrals of similar form representing $\langle A_j A_l^*\rangle$, $\langle A_j^* A_l\rangle$ and $\langle A_j^* A_l^*\rangle$.

The ranges of integration for each component of $\boldsymbol{r}_j$ and $\boldsymbol{r}_l$ are $(-X, X)$. It can be shown (e.g. Reif 1965) that as $X \to \infty$ (A.3) has a limit. This is

$$\frac{1}{(4\pi^2)^2} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \rho(\Delta \boldsymbol{r}, \Delta z) \exp\{-\mathrm{i}[\boldsymbol{R} \cdot (\boldsymbol{\mu}_j + \boldsymbol{\mu}_l) + \tfrac{1}{2}\Delta\boldsymbol{r} \cdot (\boldsymbol{\mu}_j - \boldsymbol{\mu}_l)]\}\, \mathrm{d}^2\boldsymbol{R}\, \mathrm{d}^2\Delta\boldsymbol{r} \tag{A.4}$$

where

$$R = \tfrac{1}{2}(\boldsymbol{r}_j + \boldsymbol{r}_l), \qquad \Delta\boldsymbol{r} = \boldsymbol{r}_j - \boldsymbol{r}_l \qquad \text{and} \qquad \Delta z = z_j - z_l. \tag{A.5}$$

Insertion of ρ from (3) gives

$$\langle A_j A_l\rangle = \tfrac{1}{4} r_0^3 \pi^{-1/2} \delta^2(\boldsymbol{\mu}_j + \boldsymbol{\mu}_l) \delta(z_j - z_l) \exp(-\tfrac{1}{4}\boldsymbol{\mu}_j^2 r_0^2) \tag{A.6}$$

where

$$\delta^2(\boldsymbol{a}) = \delta(a_x)\delta(a_y) \qquad \text{for} \qquad \boldsymbol{a} = (a_x, a_y). \tag{A.7}$$

Since (A.6) is real it is also the value of $\langle A_j^* A_l^*\rangle$. $\langle A_j A_l^*\rangle$ and $\langle A_j^* A_l\rangle$ may be calculated by a similar method and are found to be

$$\langle A_j A_l^*\rangle = \langle A_j^* A_l\rangle = \tfrac{1}{4} r_0^3 \pi^{-1/2} \delta^2(\boldsymbol{\mu}_j - \boldsymbol{\mu}_l) \delta(z_j - z_l) \exp(-\tfrac{1}{4}\boldsymbol{\mu}_j^2 r_0^2). \tag{A.8}$$

Appendix 2. The effect of double scatters

Consider a contribution to the diagrammatic expansion of a moment, composed solely of scatter pairs of types (i) and (ii), that is, having no double scatters. Introduce a double scatter into one field at some position between existing scatters of that field at z_a and z_b as in figure 8. After the scatter at z_a the field under consideration has a wavefront with spatial frequency

$$\boldsymbol{\mu} = \sum \boldsymbol{\mu}_i \tag{A.9}$$

where the summation is taken over all previous scatters of that field. In the absence of the double scatter a factor

$$\exp\left(-\frac{\mathrm{i}}{2k}(z_b - z_a)\boldsymbol{\mu}^2 \right) \tag{A.10}$$

would come into the integral and the field just before z_b would still have spatial frequency $\boldsymbol{\mu}$.

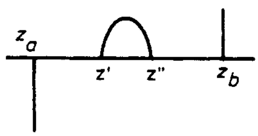

Figure 8. A double scatter inserted into a general position in a diagram.

When the double scatter is introduced we get instead of (A.10) a factor

$$(\mathrm{i}kn_1)^2\left\langle \exp\left(-\frac{\mathrm{i}}{2k}(z' - z_a)\boldsymbol{\mu}^2 \right) A(\boldsymbol{\mu}', z') \exp\left(-\frac{\mathrm{i}}{2k}(z'' - z')(\boldsymbol{\mu} + \boldsymbol{\mu}')^2 \right) \right.$$
$$\left. \times A(\boldsymbol{\mu}'', z'') \exp\left(-\frac{\mathrm{i}}{2k}(z_b - z'')(\boldsymbol{\mu} + \boldsymbol{\mu}' + \boldsymbol{\mu}'')^2 \right) \right\rangle \mathrm{d}z'\, \mathrm{d}z''\, \mathrm{d}^2\boldsymbol{\mu}'\, \mathrm{d}^2\boldsymbol{\mu}'' \tag{A.11}$$

and the field immediately before z_b has spatial frequency $\boldsymbol{\mu} + \boldsymbol{\mu}' + \boldsymbol{\mu}''$. When we perform the assembly average (A.11) becomes:

$$\tfrac{1}{4}(\mathrm{i}kn_1)^2 r_0^3 \pi^{-1/2} \delta^2(\boldsymbol{\mu}' + \boldsymbol{\mu}'')\delta(z' - z'') \exp(-\tfrac{1}{4}\boldsymbol{\mu}'^2 r_0^2) \exp\left(-\frac{\mathrm{i}}{2k}(z_b - z_a)\boldsymbol{\mu}^2 \right)$$
$$\times \exp\left(-\frac{\mathrm{i}}{2k}[(z'' - z')(\boldsymbol{\mu}'^2 + 2\boldsymbol{\mu}\cdot\boldsymbol{\mu}') + (z_b - z'')(\boldsymbol{\mu}'^2 + \boldsymbol{\mu}''^2 + 2\boldsymbol{\mu}\cdot\boldsymbol{\mu}' \right.$$
$$\left. + 2\boldsymbol{\mu}\cdot\boldsymbol{\mu}'' + 2\boldsymbol{\mu}'\cdot\boldsymbol{\mu}'')] \right) \mathrm{d}z'\, \mathrm{d}z''\, \mathrm{d}^2\boldsymbol{\mu}'\, \mathrm{d}^2\boldsymbol{\mu}''. \tag{A.12}$$

This is to be integrated over z' from z_a to z'', over z'' from z_a to z_b, and over the whole of the $\boldsymbol{\mu}'$, $\boldsymbol{\mu}''$ planes. The result is

$$-\tfrac{1}{2}k^2 n_1^2 r_0 \pi^{1/2}(z_b - z_a) \exp\left(-\frac{\mathrm{i}}{2k}(z_b - z_a)\boldsymbol{\mu}^2 \right). \tag{A.13}$$

The spatial frequency of the wave at z_b is $\boldsymbol{\mu}$ since $\boldsymbol{\mu}' = -\boldsymbol{\mu}''$. Thus the only effect of the double scatter is a factor

$$-\tfrac{1}{2}k^2 n_1^2 r_0 \pi^{1/2}(z_b - z_a). \tag{A.14}$$

This is now summed over all possible intervals (z_a, z_b) in which the scatters may occur giving a factor

$$-\tfrac{1}{2}k^2 n_1^2 r_0 \pi^{1/2} z \equiv -\tfrac{1}{2}l. \tag{A.15}$$

If we insert a second double scatter into the same field we multiply by this factor again, then divide by two to avoid double counting. Insertion of n double scatters is accomplished by multiplying by $(-\tfrac{1}{2}l)^n/n!$ and when we sum over all possible numbers of double scatters (within the one field) the answer is a factor

$$\exp(-\tfrac{1}{2}l) \tag{A.16}$$

which multiplies the contribution of the diagram in the absence of any double scatters.

Finally we allow the double scatters to occur in any field. The basic contribution is now multiplied by a factor (A.16) for each of the fields involved, leading to a factor

$$\exp[-\tfrac{1}{2}(\alpha+\beta)l]. \tag{A.17}$$

This factor accounts for the exponential decay of the mean field.

Appendix 3. An answer to a possible objection

Consider the contributions to a moment $R_{\alpha\beta}$ at a depth z. There will be a number N_{eff} such that diagrams whose orders are near to N_{eff} will represent the most important contributions to the moment, and N_{eff} increases with z. In fact, the number of scatters suffered by the field in traversing a slab of medium of thickness z obeys a Poisson distribution with a mean proportional to z (that is, scattering is a Poisson process). Thus N_{eff} is proportional to z, say

$$N_{\text{eff}} = \lambda z. \tag{A.18}$$

Since the number of possible diagrams of order N_{eff} is

$$[(\alpha+\beta)(\alpha+\beta-1)/2]^{N_{\text{eff}}} \tag{A.19}$$

and increases rapidly with N_{eff} and since the number of non-dominant contributions increases much faster than the number of dominant contributions, it may be asked whether, as z tends to infinity, the greater number of non-dominant diagrams may not outweigh their smaller effect. Since the dominant diagrams need only be one order of z greater this seems possible.

The author knows of no rigorous proof that it is not so but puts forward the following as a plausibility argument.

It is true that some non-dominant diagrams will be only one or two orders of z below the dominant diagrams of equal N, but there are relatively few of these, of order of, or less than, the number of dominant diagrams. As N increases the large majority of new diagrams created will be extremely complicated with many interlinkings. The matrices will have a large number of non-zero elements and their determinants will be completely or very nearly 'saturated', that is, will have terms in or near z^N. The associated diagrams will thus be of order z^{-N} compared with corresponding dominant diagrams, and their joint contribution will be of order

$$\left(\frac{(\alpha+\beta)(\alpha+\beta-1)}{2z}\right)^N \tag{A.20}$$

since there are about $[(\alpha+\beta)(\alpha+\beta-1)/2]^N$ of them. This tends to zero as N tends to infinity provided z is sufficiently large.

Appendix 4. The limit of $\mathbf{M}^{-1}$ as z tends to infinity

$\mathbf{M}$ may be written ((28))

$$\mathbf{M} = \tfrac{1}{4}r_0^2(\mathbf{I} + 2\mathrm{i}\zeta\mathbf{M}') \tag{A.21}$$

in which the matrix $\mathbf{M}'$ is not dependent on z. Let $\mathbf{K}$ be the matrix whose columns are the eigenvectors of $\mathbf{M}'$ so that $\mathbf{K}^{-1}\mathbf{M}'\mathbf{K}$ is a diagonal matrix Λ where

$$\Lambda_{jl} = \delta_{jl}\lambda_j.$$

Then

$$\mathbf{K}^{-1}\mathbf{M}\mathbf{K} = \tfrac{1}{4}r_0^2(\mathbf{I} + 2\mathrm{i}\zeta\mathbf{K}^{-1}\mathbf{M}'\mathbf{K}) = \tfrac{1}{4}r_0^2(\mathbf{I} + 2\mathrm{i}\zeta\Lambda). \tag{A.22}$$

This matrix is diagonal and is inverted by simply inverting each diagonal element:

$$\frac{\delta_{jl}}{\frac{1}{4}r_0^2(1+2\mathrm{i}\zeta\lambda_j)} = (\mathbf{K}^{-1}\mathbf{M}\mathbf{K})_{jl}^{-1} = (\mathbf{K}^{-1}\mathbf{M}^{-1}\mathbf{K})_{jl}. \tag{A.23}$$

Hence

$$(\mathbf{M}^{-1})_{jl} = \sum_{m,n} 4\frac{K_{jm}\delta_{mn}K_{nl}^{-1}}{r_0^2(1+2\mathrm{i}\zeta\lambda_m)} = \sum_m 4\frac{K_{jm}K_{ml}^{-1}}{r_0^2(1+2\mathrm{i}\zeta\lambda_m)}. \tag{A.24}$$

Since the matrices $\mathbf{K}$ and Λ are independent of z, each element of $\mathbf{M}^{-1}$ is a sum of constant terms or terms in ζ^{-1}. Hence as ζ tends to infinity they tend to finite limits.

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